Bridging the gap between abstract math and reality

Pavel Boytchev
boytchev@fmi.uni-sofia.bg
Faculty of Mathematics and Informatics, Sofia University

Abstract

Digital visualization is a relatively new concept, compared to others that are known to people for centuries. This paper presents an attempt to convert the mathematical concept of conic sections into several software applications designed and implemented by the author, which utilize virtual reality. The goal is to represent the basic properties of conic sections in a way that non-mathematician can understand. The applications cover different perspectives that correspond to a multidisciplinary approach. The paper describes how conic sections can be generated at home using objects from our everyday life; how to construct mechanical devices that draw conic sections; and where to find conic sections in artistic representation of the Mandelbrot set fractal.

Keywords: conic sections, visualization, multidisciplinary approach

Introduction

People know and use conic sections for a long time. They have seen these curves in many situations – the parabolic trajectory of a thrown stone, the circular concentric waves when it falls in calm water; the elliptical shadows of our heads during sunsets. Conic sections were and still are one of the most favourite objects of mathematical study and education. Students spend hours in the classroom working with circles, ellipses, parabolas and hyperbolas. They are presented with a concentrated view about these curves, a view that has been distilled for hundreds of years.

Although mathematically correct, this view may not lead to complete rationalization, because it might be hard for students to project mathematical ideas into something more comprehensible from their everyday life. A preliminary informal inquiry showed that it is a matter of some difficulty for students to identify conic sections in a non-classroom environment. This triggered the creation of visualization tools that could introduce these curves from various perspectives and these tools are the focus of this paper.

Traditionally, conic sections are described as intersections of a plane and a cone (Downs, 1979). Unfortunately, this approach does not always create transferable knowledge – i.e. knowledge that a person can use to bridge concepts from two distinct disciplines (Leitch, 2010). Some educators are forced to make compromise by choosing only few aspects of the curves, those that "have important applications in the real world" (Demana et al., 2000). A few educational materials based on this traditional approach are being advertised as "the perfect set for teaching a unit on conic sections" (Nasco, 2010).

The introduction of Dynamic Geometry Software (DSG) added a lot of expressiveness to representing conic sections and provided a playground for interesting explorations. However, DGS is still visually bound to the mathematical representation – they use the conventional geometrical visual primitives that does not relate directly to life outside the educational environment. Even advanced DGS tools like Cabri (Schumann, 2005) and the

Geometer's Sketchpad (Scher, 2010) represent conic sections in way that is just mathematically correct. The author's own attempt to combine DGS with virtual reality also did not provide any hopes for transferable knowledge (see Figure 1 for a snapshot of an interactive 3D application for experimenting with the traditional approach). It does not help students to resolve practical challenges like:

- Make an ellipse with a reading lamp
- Construct a rolling mechanism that generates an ellipse
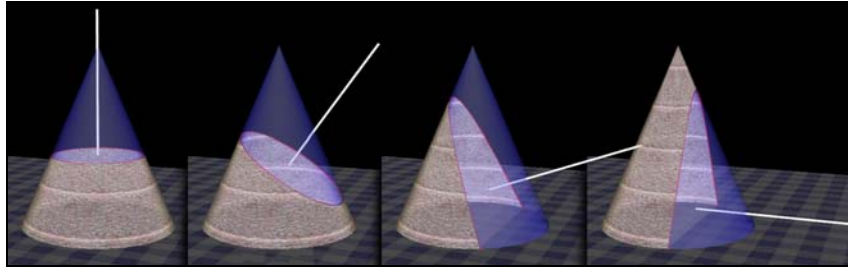- Draw a hyperbola using a fixed-length thread

Figure 1. Interactive 3D application for experimenting with the traditional representation of conic sections

Thus the main idea that initiated the work presented in this paper was to create new tools and models that describe and utilize various properties of conic section in a way that is not easy to implement in a traditional hard-copy textbook and in a contemporary DGS.

Homemade conic sections with light

In 2009 the author completed an artistic project for making an exhibition based on computer generated images and digital photographs (Boytchev, 2009). All posters featured fragments of the Mandelbrot set fractal accompanied by artistic interpretations. One of the posters features an area that resembles a coordinate system with a pair of hyperbolas and their asymptotes (Figure 2).
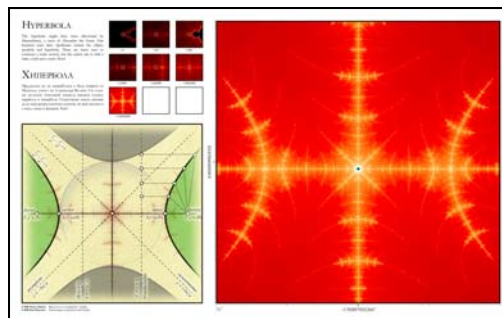

Figure 2. Hyperbolic shapes in the Mandelbrot set fractal

The description of the poster says: *The hyperbola might have been discovered by Menaechmus, a tutor of Alexander the Great. One hundred years later Apollonius named the ellipse, parabola and*

hyperbola. There are many ways to construct a conic section, but the easiest one is with a table, a ball and a torch. How?

The question raises an interesting problem. Is it possible to model all conic sections at home, using only objects from our everyday life? To answer this question the author implemented a set of interactive 3D applications within the European project InnoMathEd – Innovations in Mathematics Education on European Level (InnoMathEd, 2010). The project addresses pupils' mathematical understanding, use of ICT and competences for lifelong learning (Bianco, 2009).

The very first interactive conic section application for InnoMathEd implemented the traditional approach that involves a cone and an intersecting plane. Snapshots of this program are shown in Figure 1. Although the software provides an intuitive and easy-to-understand way for describing conic sections, it does not help students reimplement the model in the real life. This inspired the author to design other applications that are both mathematically correct and can be done at home, which is important if we want to have mathematical knowledge transferred and applicable outside the classroom.

The first group of applications used just a torch and a flat surface. The light from the torch forms the cone, while the flat surface is the intersecting plane. The image appearing on the surface is the intersection of the light cone and the plane. It can be circle, ellipse, parabola or hyperbola. It is straightforward to create a circle or an ellipse, see Figure 3 (left), but other conics extend to infinity. Is it still possible to generate them? Figure 3 (right) shows a side view of the virtual model. If the upper side of the light cone is parallel to the table, then we get a parabola and if it points above the horizon – a hyperbola. The proof that this approach generates conic sections is beyond the scope of the paper; however, this setup can be translated into the traditional representation of a cone and an intersecting plane.
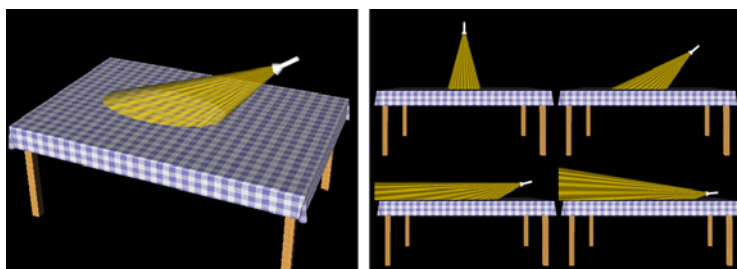

Figure 3. Making an ellipse (left) and all conic sections (right) using a torch
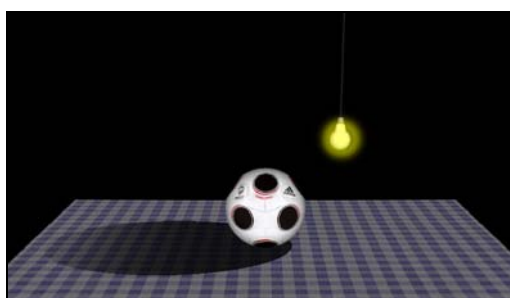

Figure 4. Making an ellipse using shadows

If we use a light source that emits in all directions, we can still model conics, by generating shadows with any spherical object. An elliptical shadow generated by another interactive 3D application is shown in Figure 4.

By moving the light, we can generate the other conic sections. Parabola appears when the bulb is at the same level as the upper side of the ball, hyperbola – when it is below it. The position of the ball has an important mathematical meaning. The point of contact is a focal point (focus) and the ball is a Dandelin sphere (Kendig, 2005; Weisstein, 2010).

There is a third interactive 3D application, which uses just a tube. If someone looks through it, the cone of sight will "cut" a conic section from the ground. The type of the section depends on whether the horizon is seen through the tube.

Virtual models of mechanical devices

Except for the applications using the light or the absence of light to model conic sections, the author has implemented a rich set of non-interactive 3D applications that represent models of mechanical devices drawing conic curves (Boytchev, 2010a). These devices represent various methods and could foster interesting mathematical research activities for students. All devices are built by a set of primitive elements including a pencil that draws on paper.

It is easy and straightforward to model a device drawing a circle, however we should think of a more enhanced device if we need all conic section. Figure 5 presents snapshots of such device where the pencil can slide forward and backward controlled only by gravity. Some of the mathematical problems occurred during the design of these animations were:

- How long should the pencil be?
- How to model the infinite parabolas and hyperbolas?
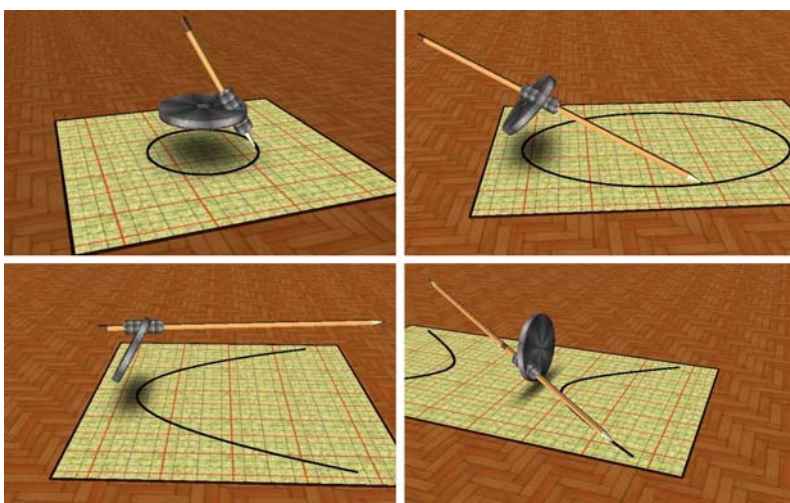- How to draw both branches of hyperbolas?

Figure 5. Pencil attached to a disk can draw any conic section

The model represented in Figure 5 recreates the conical nature of the conic sections – the pencil rolls on the surface of the cone and the paper acts as an intersecting plane. The

gravity forces the pencil to slide forward as much as needed to reach the paper. The disk rotation used in the model is essential for the formation of the conic curves, but is not the only way to make them with rotation.

The sum of two vectors with different lengths rotating at the same angular speed but in opposite directions a vector traverses an ellipse; see Figure 6 (left). The two shorter beams represent one of the vectors; the longer ones represent the other. The construction uses a parallelogram to maintain mechanical stability and to demonstrate the commutative nature of vector addition. Figure 6 (right) represents a ruled surface called hyperboloid. It is created by a tilted line rotated around a vertical line. The intersection of the hyperboloid and a horizontal plane is a circle, while the intersection with a vertical plane is a hyperbola.
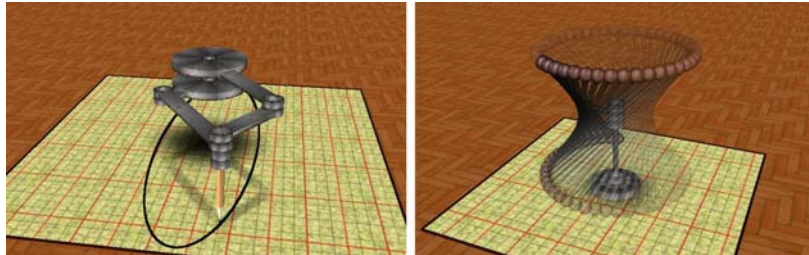

Figure 6. Other rotational methods to generate some conic sections

The models shown in Figure 5 and Figure 6 are not quite practical. For example, it is hard to use these devices to make elliptical windows. Carpenters have solved this problem by using a simple yet effective device called the Trammel of Archimedes (Apostol & Mnatsakanian, 2009). This device can draw an ellipse with predefined major and minor axes. The device provides sufficient precision for carpentry.

Variations of the trammel are shown in Figure 7. A fixed length segment slides along two perpendicular pair of rails and the pencil is attached near segment's midpoint. If the pencil is exactly in the midpoint, it will draw a circle. An ellipse is produces if the pen is shifted away from the midpoint as shown in Figure 7. In reality, the devices used by carpenters are slightly different. For example, the pen is often attached "outside" the sliding segment's endpoints. This allows the carpenters to use much shorter rails.
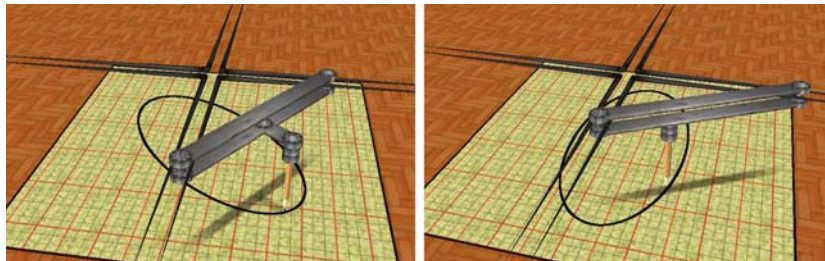

Figure 7. Variation of Trammel of Archimedes

Like the device based on two rotating vectors shown in Figure 6 (left), the Trammel of Archimedes can draw any curve starting from a straight line and ending with a circle traversing ellipses with various eccentricities. Unfortunately, the trammel cannot draw a

parabola or a hyperbola. There is another ancient method of drawing ellipses utilizing the property that the sum of distances from each point on the ellipse to its foci is constant.

Figure 8 contains snapshots of four 3D animations visualizing the four types of curves generated with a fixed-length thread. The cases in the top two snapshots are rather conventional and are pictured just for completeness. An interesting challenge is to design a similar mechanism for parabolas and hyperbolas, because most hard copy and online textbook show only the case with an ellipse.
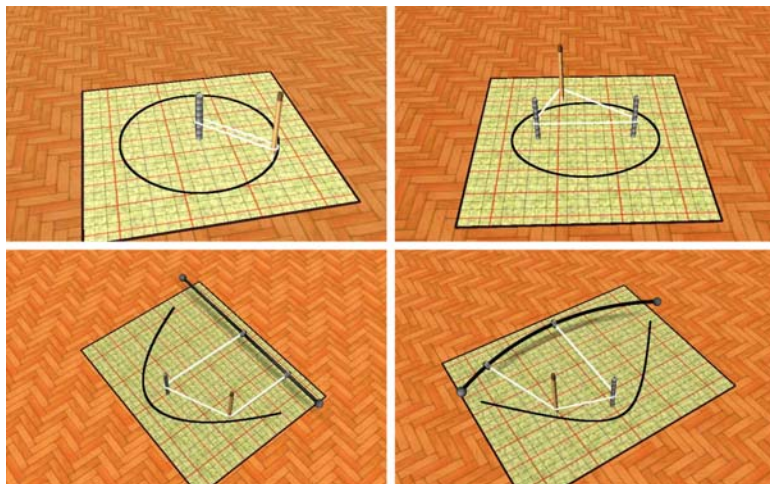

Figure 8. A thread used to draw conic sections

The lower left snapshot shows a parabola. The thread has its both ends freely attached to a rail that collinear to the parabola's directrix. These ends can slide along the rail keeping the thread perpendicular to the rail. The midsection of the thread embraces the pen and the focus point. A similar device is shown in the last snapshot in Figure 8. It has the same structure, except that the rail is a circular arc, not a straight line. In this case, the pencil draws a hyperbola. It is a nice mathematical exercise to prove that the curve is really a hyperbola.

The topic of modelling the construction of conic intersections is virtually unlimited. The last example is to use a device drawing a hypotrochoid. Figure 9 represents such device where a disk rolls inside a ring without slipping. The radii of the disk and the ring are selected in such a way, that the attached pencil draws an ellipse.
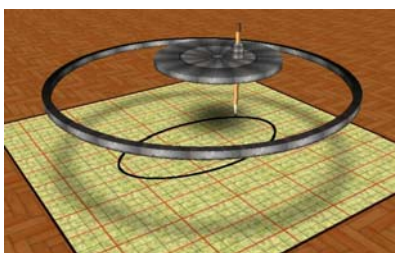

Figure 9. A hypotrochoidal ellipse

Planned student activities

The interactive 3D applications and the 3D animations presented in this paper are included in a larger set of more than 50 models of devices. Some of them draw mathematical curves, other represent mathematical transformations, construction of 3D surfaces, and even non-geometrical phenomena like normal distribution in statistics.

All these applications are rather new and they are still not used in the classroom or at home. Teaching materials are now being planned that will utilize the full multidisciplinary power of the models. The rest of the section describes briefly some potential activities:

- *Computer Science/Computer Graphics*. Each model is a program written in a programming language. The design and the implementation of such program require specific skills and knowledge. Some models are very simple from programming point of view, others are quite complex. The building of precisely selected series of models can improve these skills and knowledge.

- *Physics/Mechanics*. The virtual models included in the collection are not just mathematical abstractions. They comply with known physical restrictions and utilize the properties of elements made of different materials. A thread can change its shape, while a solid beam cannot. Some of the models are rather tricky to design. For example, the device in Figure 6 (left) has a carefully chosen vertical layering of elements to eliminate physical collision.

- *Mathematics/Geometry*. There are many papers and web sites describing the properties of conic sections. However, the construction of devices requires understanding these properties at a higher level of abstraction. Converting a mathematical model into mechanical one is a real mathematical and engineering challenge. Alternatively, mathematical challenges are hidden in the models themselves. They can be used to teach mathematical concepts and be used as a experimental playground for various tasks.

- *Art*. The collection of animations inspired the author to create a short mathematical film that illustrates seven different ways of constructing ellipses (Boytchev, 2010b). Although purely artistic, the making of the film was based on solving many geometrical, mechanical and programming challenges. Students may also be engaged is similar multidisciplinary activities where they blend scientific and artistic designs constructing an artefact of their choice.

The InnoMathEd project is a nice opportunity to create teaching/learning activities and evaluate them in classroom and home use. The adoption of techniques from virtual reality and gaming industry makes the animations and the applications more appealing.

Conclusion

The set of virtual models described in this paper is still under construction. There are many other mathematical concepts that are still not implemented. Although the models are not tested in real educational environment, it is expected that they will provide a foundation of a new and effective approach to learning. The ground of this claim is based on author's and his colleagues' experience with designing and implementing other interactive 3D virtual environments, like the set of ten DALEST (Developing an Active Learning Environment for Stereometry) applications (DALEST, 2008).

The software by itself does not imply that it is effective for educational purposes, because technology in education is educationally neutral. Whether its use is effective or not in the classroom, depends on how it is used. In this respect, the presented software provides educational perspectives that are not feasible in the traditional mathematical textbooks and are not used by the modern dynamic geometry software. Whether these perspectives will be utilized, is related to the actual application in the classroom.

The major expected benefits of using the presented models in mathematical, physical and computer science education are:

- to provide a comprehensive set of different view points;
- to represent conic curves in a way that can be recreated at home;
- to display the main properties of conics in a clear way;
- to build an engaging and entertaining virtual environment;
- to utilize the power of ICT in areas where conventional textbooks fail.

All models are available free to the public both as source codes and as 30-seconds video clips. The clips are included in the deliverables of InnoMathEd project. Additionally, most of the clips were already shown to freshmen university students in their Geometry of Motion course.

Acknowledgements

References

Apostol, T., & Mnatsakanian, M. (2009). A new look at the so-called trammel of Archimedes. *American Mathematical Monthly*, 116(2).

Bianco, T. (2009). *Needs analysis report for innomathed - innovations in mathematics education*. University of Augsburg.

Boytchev, P. (2009). *Exhibition "Seduction"*. Retrieved 30 April 2010 from http://mandelbrot-set.elica.net

Boytchev, P. (2010a). *Mathematical devices playlist at ElicaTeam's Channel*. Retrieved 30 April 2010 from http://www.youtube.com/elicateam#g/c/6534E936D46257BF

Boytchev, P. (2010b). *Film "Ellipses…"*. Retrieved 11 July 2010 from http://www.youtube.com/watch?v=1v5Aqo6PaFw

DALEST (2008). *Elica-DALEST applications*. Retrieved 30 April 2010 from http://www.elica.net/site/museum/Dalest/dalest.html

Demana, F., Waits, B., Foley, G., & Kennedy, D. (2000). *Precalculus. Functions and Graphs*. Addison Wesley Longman.

Downs, J. (1979). *Practical Conic Sections: The Geometric Properties of Ellipses, Parabolas and Hyperbolas*. Dover Publications.

InnoMathEd. (2010). *InnoMathEd*. Retrieved 30 April 2010 from http://www.math.uni-augsburg.de/prof/dida/ innomath

Kendig, K. (2005). *Conics*. Cambridge University Press.

Leitch, P. (2010). *Conic Sections*. The North Dakota Curriculum Initiative. Retrieved 11 July 2010 from http://ndcurriculuminitiative.org/index.php/lesson_bank/lesson/conic_sections/

Nasco (2010). *Conic Sections Set ~ Conic Sections*, Product web page. Retrieved 11 July 2010 from http://www.enasco.com/product/TB23712T

Scher, D. (2010). *Exploring Conic Sections with The Geometer's Sketchpad (Version 4)*, Key Curriculum Press.

Schumann, H. (2005). *Introduction to Conics with Cabri 3D*. EduMath 20 (6/2005).

Weisstein, E. (2010). *Dandelin Spheres*. From MathWorld – A Wolfram Web Resource. Retrieved 30 April 2010 from http://mathworld.wolfram.com/DandelinSpheres.html